

**Interplay between impurities and correlations in superconducting nanorings**

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The properties of nanoscopic rings with electronic correlations and impurities are analyzed numerically by means of two methods. First, we carry out exact diagonalization of one-dimensional rings that consist of up to several lattice sites. Then, we perform the Bogoliubov–de Gennes equation studies of finite-width rings consisting of a few hundred sites. Results obtained from both approaches are shown to be consistent. We demonstrate how the system properties are affected by various configurations of impurities for both repulsive and attractive electron–electron interactions. In the case of attractive interaction we show that the nanoscopic properties are mainly determined by the competition between tendencies toward pairing and formation of the density waves. Since the impurities act as pinning centers for the density waves, their configuration determines the result of this competition.

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**I. INTRODUCTION**

Recent developments in the fabrication techniques give rise to intensive investigations of the nanoscopic regime, where the physical properties of the system strongly depend on its size. Usually, in theoretical approaches Coulomb correlations cannot be taken into account exactly. In many cases a coupling between the nanosystem and macroscopic leads introduces additional serious complications. From this point of view nanorings are very attractive, since their properties can be investigated without such a coupling and exact results can be obtained. One of the most interesting features of small metallic rings is the presence of persistent currents.<sup>1,2</sup> The currents flow along the rings in equilibrium state, when a constant external magnetic field is applied. Such currents are observed in many experiments, however their magnitudes are much larger than those predicted theoretically.<sup>3</sup> This discrepancy still remains an open problem. It might be attributed to the influence of electronic correlations magnetic impurities,<sup>4</sup> or even superconducting fluctuations. If only one of these effects is taken into account, the problem seems to be tractable and many theoretical predictions have already been obtained. Generally, it is believed that both impurities as well as electronic correlations reduce the persistent current. On the other hand, there are some indications, that the persistent current can increase when both these effects are present. In particular, it has been shown that in the random potential, two interacting electrons can propagate coherently on a much larger distance than the one-particle localization length.<sup>5</sup> In some cases, the disorder may lead to  $h/2e$  energy levels periodicity, whereas the corresponding eigenfunctions exhibit a pairing effect.<sup>6,7</sup> The localization length itself depends on the electronic correlations, decreasing (increasing) for repulsive (attractive) interactions.<sup>8</sup> For a finite disorder, the persistent currents in the system with repulsive interactions are larger than those in the system with attractive ones.<sup>9</sup> This is because local-density fluctuations are reduced in the presence of repulsive interactions. It is also known that such counterintuitive cooperation of correlation and disorder can also take place in macroscopic systems of higher dimensionality.<sup>10</sup> The above argument clearly demonstrates

that the interplay between disorder and electronic correlations is of crucial importance in these systems and further investigations are needed.

For many years, the investigated rings have been made out of normal metals. Only recently, has technological progress allowed investigation of superconducting nanowires.<sup>11</sup> As the sizes of such systems can be comparable to the coherence length, a question concerning the possible onset of superconductivity became very interesting. The experiments show that, in sufficiently thin nanowires, the superconductivity does not occur.<sup>12</sup> The suppression of superconductivity is usually attributed to the destruction of the phase coherence by quantum phase slips.<sup>13,14</sup> The spatial confinement originating from the geometry of a nanowire is responsible for an inhomogeneity of the superconducting order parameter.<sup>15</sup> The physical properties of mesoscopic superconducting rings are presently intensively investigated. For extremely type II superconductors fabrication of nanorings should be possible and will probably be the subject of future experiments. This problem contains interesting physics, because both superconducting and one-electron persistent currents may occur in such systems. Moreover, one may expect that phenomena typical for low-dimensional correlated systems, e.g., charge-density waves (CDW), may be present as well. On the other hand, magnetic flux strongly affects the CDW ground state in ring-shaped systems<sup>16</sup> and may even lead to its destruction.<sup>17</sup> The CDW order could be strongly affected also by the impurities, as they play the role of pinning centers.<sup>18</sup>

The aim of this paper is a detailed investigation of the nanoscopic rings with pairing correlations and impurities. In particular, we focus on the influence of impurities on competition between superconductivity and the CDW. In the first part of this paper, we consider rings small enough to be investigated within the exact diagonalization methods, leading to rigorous results. Then, we compare these results to the ones obtained for much larger systems with the help of the Bogoliubov–de Gennes (BdG) equations.

The outline of this paper is as follows. In Sec. II we recall main results concerning the influence of the Coulomb interactions on the persistent current in nanorings. In Sec. III and

IV we investigate nanorings with pairing interactions in the presence of impurities. In Sec. III we present rigorous results obtained from an exact diagonalization study, whereas Sec. IV contains similar results obtained with the help of BdG equations for rings containing up to a few hundred sites. Finally, in Sec. V we summarize our results.

## II. CORRELATIONS AND IMPURITIES IN NANORINGS

We start our investigations with small (up to 12 sites) rings described by the Hubbard Hamiltonian,

$$H_{\text{Hubb}} = -t \sum_{\langle i,j \rangle, \sigma} e^{i\theta_{ij}} a_{i\sigma}^\dagger a_{j\sigma} + U \sum_i n_{i\uparrow} n_{i\downarrow}, \quad (1)$$

where  $a_{i\sigma}^\dagger$  ( $a_{i\sigma}$ ) creates (annihilates) an electron on site  $i$  with spin  $\sigma$ ,  $U$  is the on-site electron-electron interaction, and  $n_{i\sigma} = a_{i\sigma}^\dagger a_{i\sigma}$ .  $t$  is the nearest-neighbor hopping integral in the absence of magnetic field ( $t > 0$ ), and  $e^{i\theta_{ij}}$  is the Peierls phase factor that describes the orbital response of the system to an external magnetic field

$$\theta_{ij} = \frac{2\pi}{\Phi_0} \int_{\mathbf{R}_j}^{\mathbf{R}_i} \mathbf{A} \cdot d\mathbf{l}, \quad (2)$$

where  $\Phi_0 = hc/e$  is the flux quantum. This Hamiltonian has exactly been diagonalized with the help of the Lanczös algorithm. It is one of the most effective computational tools for searching the ground state and some low-lying excited states of a finite system. From the ground state, we can compute all static and dynamic properties, and in this sense, we obtain a complete characterization of a model at low temperatures. At zero temperature the flux-induced current  $I$  is calculated as

$$I = - \frac{dE_0}{d\Phi}, \quad (3)$$

where  $E_0$  is the ground-state energy and  $\Phi$  is the magnetic flux piercing the ring. At finite temperature in Eq. (3) one should use the free energy  $\mathcal{F}$  instead of the ground-state energy. Unfortunately, the Lanczös method gives only a few lowest eigenenergies, and therefore the calculations are restricted to relatively low temperatures. On the other hand, for smaller systems other methods enabled us to find all the eigenenergies of the Hamiltonian, and the resulting current can be obtained for an arbitrary temperature. In Fig. 1 we demonstrate how the persistent currents are destroyed by the Coulomb repulsion (upper panel) and by an increase in temperature (lower panel). It has also been shown that the persistent currents are reduced in the presence of the thermal equilibrium noise.<sup>19</sup> These results are intuitive and well known, and therefore we will not discuss them here. They are presented only for comparison with the results discussed further.

In order to account for the presence of nonmagnetic impurities we extend the Hubbard Hamiltonian

$$H = H_{\text{Hubb}} + \sum_i w_i (n_{i\uparrow} + n_{i\downarrow}), \quad (4)$$

where  $w_i$  is the potential of an impurity at site  $i$ .

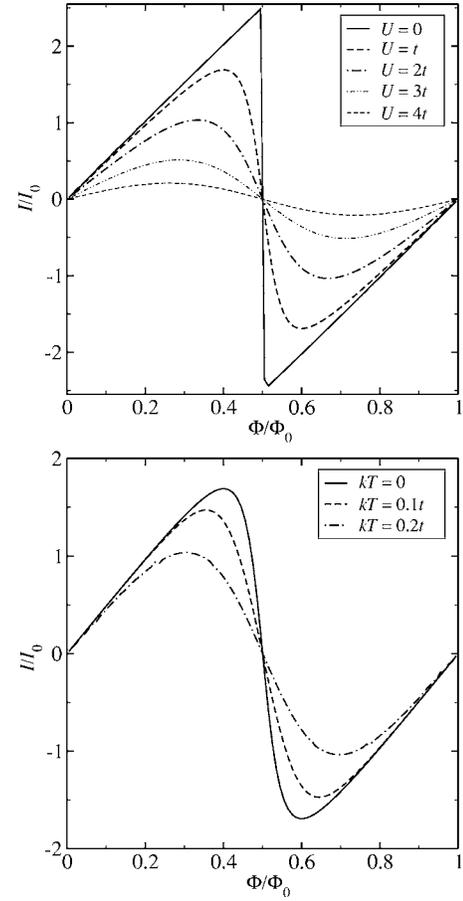


FIG. 1. Flux dependence of the persistent current for a ring containing 10 lattice sites with repulsive interaction and in the absence of impurities. Results presented in the upper panel have been obtained for  $T=0$  and various values of  $U$ , as indicated in the legend. The lower panel shows results obtained for  $U=t$  at various temperatures. We have denoted  $I_0 = t/\Phi_0$ .

We start our investigations with a single impurity, i.e.,  $w_i = \delta_{i0}W$ , in a small ring. The first question that arises in this case concerns the impact of the electron correlations and impurity on the magnitude of the persistent current. To answer this question, for a wide range of the potentials  $U$  and  $W$ , we have found a magnetic flux that produces the maximal value of the persistent current ( $I_{\text{max}}$ ). Figure 2 shows how  $I_{\text{max}}$  depends on  $U$  and  $W$  for the half-filled case, i.e., when the number of electrons is equal to the number of sites. One can see that there is a  $W \rightarrow -W$  symmetry. It is an obvious result of the particle-hole symmetry of the Hubbard model. In the case of the attractive electron-electron interaction, the maximal value of  $I_{\text{max}}$  takes place for  $W=0$ , i.e., in the absence of impurities. Contrary to this result, for repulsive on-site interaction ( $U > 0$ ),  $I_{\text{max}}$  takes on the maximal value in the presence of impurity, when  $|W|$  is slightly larger than  $U$ . It means that for a fixed value of the impurity potential, the maximum of  $I_{\text{max}}$  corresponds to the finite repulsive interaction and this result holds independently of the sign of the impurity potential. Similarly, in Ref. 9 it has been shown that in a disordered ring, for the repulsive interaction the persistent current is larger than for the attractive one. It originates

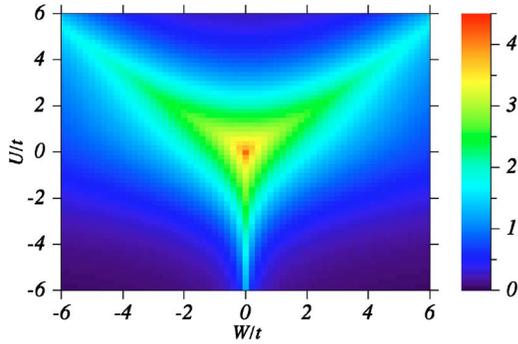


FIG. 2. (Color online) Dependence of the  $I_{\max}/I_0$  on the electron-electron interaction ( $U$ ) and the impurity potential ( $W$ ) for a 6-site ring with a single impurity.

from the fact that repulsive interaction reduces impurity-induced density fluctuations, whereas attractive interaction may lead to CDW with impurities acting as pinning centers.

One meets a much more interesting situation in a case of many impurities. In particular, a question arises whether an impurity added to the previously considered ring leads to a further enhancement or reduction of the persistent current. The answer to this question strongly depends on the relative positions of the impurities. Figure 3 shows  $I_{\max}$  as a function of the distance  $d$  between two impurities, i.e., for  $w_i = W(\delta_{i0} + \delta_{id})$ . One can see, that for two impurities located at the nearest and the next nearest neighbors the persistent current is smaller than in the case of a single impurity but larger than for  $W=0$ . Then, one can see an oscillatory character of this dependence with increasing amplitude. Namely,  $I_{\max}$  is enhanced (reduced) when the distance is an odd (even) multiple of the lattice constant. For a sufficiently large distance between impurities, the persistent current may even exceed  $I_{\max}$  obtained for the case of a single impurity. The obtained oscillatory behavior may be attributed to the density oscillations

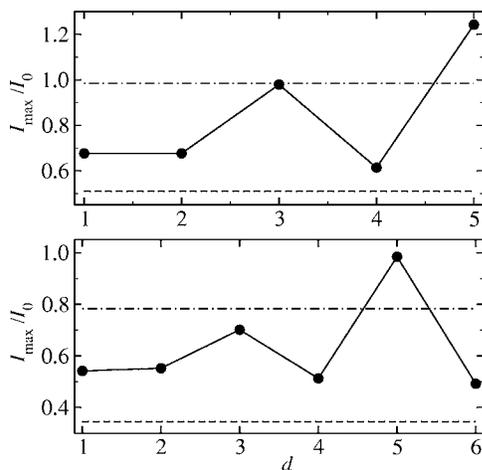


FIG. 3.  $I_{\max}$  calculated for a ring with two impurities as a function of the distance between them. We have taken  $U=3t$ ,  $W=-4t$ , and  $T=0$ . The lower (upper) panel corresponds to the 12-site (10-site) ring. In both the panels horizontal lines indicate  $I_{\max}$  in the absence of impurity (the lower one) and in the presence of a single impurity (the upper one).

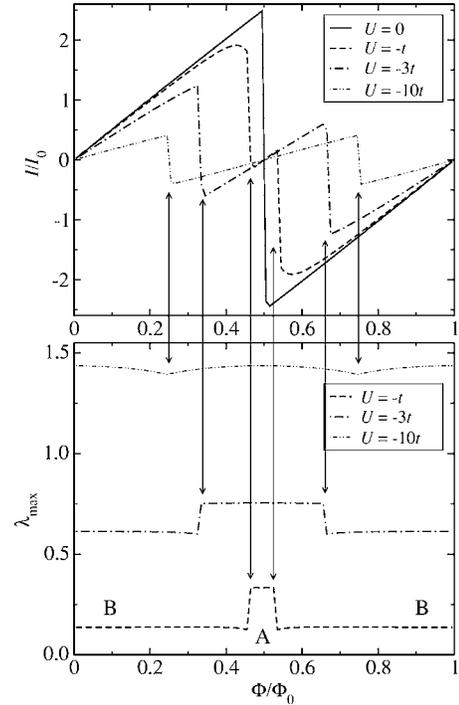


FIG. 4. The upper panel shows the flux dependence of the persistent current for a 10-site ring in the absence of impurities. The curves correspond to various values of the pairing potential. The lower panel shows the pair susceptibility for the same parameters as in the upper panel. The arrows indicate on the coincidence between the abrupt change of pair susceptibility and the reversed circulation of the persistent currents. A and B mark regimes of large and small values of  $\lambda_{\max}$ , respectively.

induced by impurities (Friedel oscillations). It is expected that these oscillations asymptotically decay with the distance  $x$  as  $\cos(2k_F x + \eta)x^{-\delta}$ , where  $k_F$  is the Fermi momentum and  $\eta$  parametrizes the interaction (see Ref. 8 for the details). Therefore, oscillations originating from different impurities may interfere. For the half-filled case,  $2k_F = \pi$  and the oscillations produced by impurities separated by odd (even) number of the lattice constants interfere destructively (constructively).

Thus far, we have focused on the case of a repulsive electron-electron interaction and have shown that the configuration of impurities is of vital importance for the magnitude of the persistent current. In the case of the attractive interaction one may expect that this effect should be even more pronounced, since in such a model a CDW instability occurs also in the absence of impurities. Then, impurities may enhance this ordering, acting as pinning centers. This problem will be investigated in Sec. III.

### III. NANORINGS WITH PAIRING INTERACTION

We start with the attractive Hubbard model without impurities. The upper panel in Fig. 4 shows the persistent current as a function of the magnetic flux for different values of the on-site pairing potential  $U$ . As the interaction increases, the system evolves toward a state where the persistent current

exhibits  $\Phi_0/2$  periodicity. Simultaneously, one may observe a reduction of the magnitude of the persistent currents when the pairing increases. The change of periodicity may be a signature of a current made out of carriers having charge  $2e$ ,<sup>20,21</sup> although one does not expect occurrence of a superconducting phase in such a small system. Similar results have recently been obtained for the boson-fermion model.<sup>22</sup> However, the change of periodicity of the persistent current is not necessarily related to the pairing interaction. In particular, for a genuinely strong on-site repulsion, the system consisting of  $N_e$  electrons shows  $\Phi_0/N_e$  and  $\Phi_0/2$  periodicities.<sup>23,24</sup> Therefore, it would be important to distinguish between the possible mechanisms that may be responsible for the change of periodicity. In order to perform this task, we calculate the pairing correlation function for local Cooper pairs. Usually, one calculates the susceptibility of the form<sup>25</sup>

$$\chi_{\text{sup}} = \frac{1}{N} \sum_{ij} (\langle \hat{\Delta}_i \hat{\Delta}_j^\dagger \rangle - \langle a_{i\uparrow} a_{j\uparrow}^\dagger \rangle \langle a_{i\downarrow} a_{j\downarrow}^\dagger \rangle), \quad (5)$$

where the Cooper pair creation operator is given by  $\hat{\Delta}_i^\dagger = a_{i\downarrow}^\dagger a_{i\uparrow}^\dagger$ . The increase of this quantity indicates that pairing correlations are enhanced. However, in the presence of magnetic field, we cannot directly use this form of the susceptibility. Instead, we need a gauge-invariant quantity. This will ensure that the susceptibility will show the same periodicity as the system under investigation. Therefore, we construct a Hermitian matrix

$$\chi_{ij} = \langle \Delta_i \Delta_j^\dagger \rangle - \langle a_{i\uparrow} a_{j\uparrow}^\dagger \rangle \langle a_{i\downarrow} a_{j\downarrow}^\dagger \rangle, \quad (6)$$

and investigate its eigenvalues. They are gauge invariant and possess the same periodicity as the energy spectrum. In an infinite system, the superconducting instability corresponds to the divergence of  $\chi_{\text{sup}}$ . In such a case the maximal eigenvalue of  $\chi_{ij}$ ,  $\lambda_{\text{max}}$ , diverges as well. Therefore, in the presence of magnetic field we use  $\lambda_{\text{max}}$  as a quantity that probes the tendency toward the formation of the paired state. The lower panel of Fig. 4 shows  $\lambda_{\text{max}}$  as a function of magnetic field. One can see that  $\lambda_{\text{max}}$  strongly depends on the magnetic flux. This quantity is maximal for exactly the same values of the flux (regime marked A in Fig. 4), for which the persistent currents are modified by the pairing correlations. Therefore, we identify these regimes as precursors to the superconducting state. For a weak attraction, the enhancement of  $\lambda_{\text{max}}$  in these regimes is pronounced. As the interaction strength increases, these regimes become wider, however, simultaneously the field dependence of  $\lambda_{\text{max}}$  gradually vanishes. It means that for a weak interaction, superconducting correlations are enhanced by specific values of the magnetic flux, whereas for strong coupling the pairing tendency is independent of the flux. This is a remnant of the Little and Parks results obtained for macroscopic thin superconducting films.<sup>26,27</sup> This problem will be discussed in more detail in Sec. IV within the Bogoliubov–de Gennes approach. Additionally, as one may expect, the maximal value of  $\lambda_{\text{max}}$  increases when the pairing interaction becomes stronger, supporting our interpretation of this quantity.

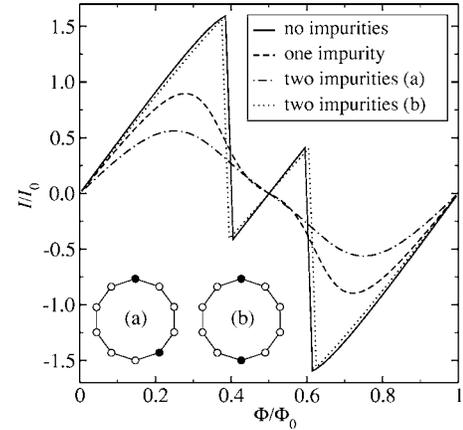


FIG. 5. Flux dependence of the persistent current for the 10-site ring with attractive interaction  $U=-2t$  and impurity potential  $W=0.5t$ . The curves have been obtained for a system without impurities, with a single impurity, and two impurities in configurations presented in the insets.

Now, we extend the analysis taking into account the impurities. It has already been shown (see Fig. 2) that here, in contradiction to the case of the repulsive interaction, a single impurity always reduces the persistent current. Playing a role of a CDW pinning center, it stabilizes density waves, which compete with pairing. It shows up as a vanishing of the regime of reverse circulation of the persistent current. However, in the presence of many impurities they can reduce as well as enhance the persistent current, depending on their configuration. Figure 5 presents the flux dependence of the persistent currents for some configurations of two impurities.

One can see that a single impurity always reduces the persistent current and destroys the tendency toward formation of the paired state. On the other hand, when an additional impurity is introduced into the system, the persistent current can be significantly larger than in the case of a single impurity. This, however, depends on the relative position of the impurities. Similarly to the case of repulsive interaction, when the distance between the impurities is an odd (even) multiple of the lattice constants, the persistent current and the pairing tendency are enhanced (reduced). This is a result of the competition between the CDW order and the formation of Cooper pairs. For the half-filled case, the electron density in the CDW state oscillates with the wave vector equal to  $\pi$ . Therefore, depending on configuration of the impurities, the density waves pinned by them can interfere constructively or destructively, increasing or reducing the CDW order. In the first case, the persistent current is less than in the presence of a single impurity and the tendency toward formation of the paired state is almost destroyed. In the latter case, the persistent current can be as large as in the clean system. One can see from Fig. 5 that for some configurations of impurities the persistent current can be almost indistinguishable from that obtained for the clean system.

## IV. NANORINGS OF FINITE WIDTH

### A. The formalism

In the preceding sections we have analyzed one-dimensional systems only consisting of several sites. This

limitation originated from the Lanczös method, which we have used to diagonalize the Hamiltonian. In this section we extend this analysis and account for finite-width rings consisting of a few hundred lattice sites. In such a case we cannot use the exact diagonalization method and, therefore, the interaction term is analyzed at the mean-field level. In particular, we decouple this term in the following way:

$$U \sum_i n_{i\uparrow} n_{i\downarrow} \approx U \sum_i (\langle n_{i\uparrow} \rangle n_{i\downarrow} + n_{i\uparrow} \langle n_{i\downarrow} \rangle) + U \sum_i \Delta_i a_{i\uparrow}^\dagger a_{i\downarrow}^\dagger + \text{H.c.}, \quad (7)$$

where the superconducting order parameter reads  $\Delta_i = \langle a_{i\downarrow} a_{i\uparrow} \rangle$ . The first term on the right-hand side of Eq. (7) is responsible for the formation of density waves. For the negative  $U$ , the second term leads to isotropic  $s$ -wave superconductivity. In the following, we assume that there is no magnetic ordering, i.e.,  $\langle n_{i\uparrow} \rangle = \langle n_{i\downarrow} \rangle = \bar{n}_i$ . As the system under investigation is inhomogeneous, both the superconducting and CDW order parameters are site dependent and have to be determined in a self-consistent manner from the Bogoliubov–de Gennes (BdG) equations.<sup>27</sup> This approach has most commonly been used for the investigation of the vortex structure in macroscopic superconducting systems.<sup>28–30</sup> We introduce a set of fermionic operators  $\gamma_{n\sigma}^{(\dagger)}$

$$a_{i\uparrow} = \sum_l u_{il} \gamma_{l\uparrow} - v_{il}^* \gamma_{l\downarrow}^\dagger,$$

$$a_{i\downarrow} = \sum_l u_{il} \gamma_{l\downarrow} + v_{il}^* \gamma_{l\uparrow}^\dagger,$$

where

$$\sum_j \begin{pmatrix} \mathcal{H}_{ij} & U \Delta_i \delta_{ij} \\ U \Delta_i^* \delta_{ij} & -\mathcal{H}_{ij}^* \end{pmatrix} \begin{pmatrix} u_{jl} \\ v_{jl} \end{pmatrix} = E_l \begin{pmatrix} u_{il} \\ v_{il} \end{pmatrix}. \quad (8)$$

Here, the single-particle Hamiltonian is given by

$$\mathcal{H}_{ij} = -t \delta_{i+\delta_j} e^{i\theta_{ij}} + (U \bar{n}_i + w_i - \mu) \delta_{ij}, \quad (9)$$

where  $\mu$  is the chemical potential. The superconducting order parameter is determined self-consistently by

$$\Delta_i = - \sum_l u_{il} v_{il}^* \tanh\left(\frac{E_l}{2kT}\right). \quad (10)$$

Also, the local electron concentration is calculated self-consistently in the following way:

$$\bar{n}_i = \sum_l |u_{il}|^2 f(E_l) + |v_{il}|^2 f(-E_l), \quad (11)$$

where  $f$  is the Fermi distribution function. This quantity allows one to define the CDW order parameter

$$\Omega_i = (-1)^i (\bar{n}_i - \bar{n}), \quad (12)$$

where  $\bar{n} = 1/N \sum_j \bar{n}_j$  is the average concentration of electrons in the ring. Up to this point, we have investigated total current flowing along the one-dimensional ring. Now, we investigate the current distribution in a ring of a finite width. We follow the procedure described in Ref. 31. Namely, the cur-

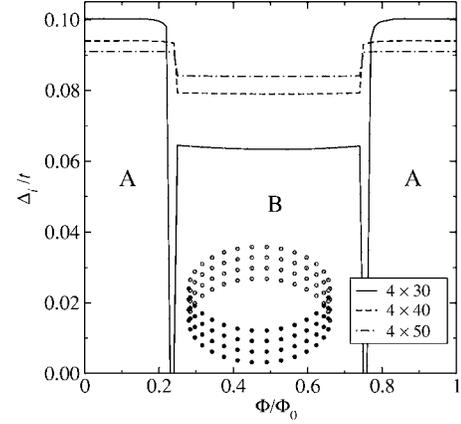


FIG. 6. Flux dependence of the superconducting order parameter for nanorings of various sizes without impurities.  $U = -1.25t$  and  $kT = 0.01t$  have been used. We have marked regions A and B (see text). As an example, in the inset we show the geometry of the  $4 \times 30$  ring.

rent from site  $i$  to the neighboring site  $j$  reads  $I_{ij} = \langle \partial H / \partial \bar{A}_{ij} \rangle$ , where  $\bar{A}_{ij}$  is the integral of the vector potential between sites  $i$  and  $j$ . Then, it is easy to show that

$$I_{ij} = - \frac{2et}{\hbar c} \text{Im} \left[ e^{i\theta_{ij}} \sum_l \tanh\left(\frac{E_l}{2kT}\right) (v_{il} v_{jl}^* - u_{il}^* u_{jl}) \right]. \quad (13)$$

## B. Numerical results for a clean system

We have solved the BdG equations for rings of sizes  $4 \times M$ , where  $M = 30, 40, 50$ . The notation  $N \times M$  means that the ring consists of  $N$  sites along the width and  $M$  sites along the circumference, i.e., the ring is made out of a rolled  $N \times M$  stripe (see the inset in Fig. 6). In order to determine how the properties of the ring depend on its size, we have started our investigations with a system without impurities. Because of the small size of the system the influence of its edge is nonnegligible and results in an inhomogeneity of  $\Delta_i$ .<sup>15</sup> However, the differences of flux dependence of  $\Delta_i$  between various lattice sites are of quantitative character only. Therefore, we present results for one particular site, which is close to the midway point between the ring's edges. In Fig. 6 we show the magnitude of the superconducting order parameter as a function of the applied magnetic flux at low temperature and for various  $M$ . One can see that the flux dependence of the order parameter increases with decreasing circumference of the ring. At low temperature, the magnitude of  $\Delta_i$  takes on two different values. In analogy to Fig. 4 we denote the regimes of high and low values of  $|\Delta_i|$  as A and B, respectively. Comparing Figs. 6 and 7, one can note opposite directions of the supercurrent's circulation, respectively, to the normal-state persistent current in regimes A and B. Moreover, the regime A (B) becomes wider (narrower) when the circumference increases. In the case of infinite circumference of the ring, the normal-state persistent currents vanish, regimes A and B become indistinguishable, and the system

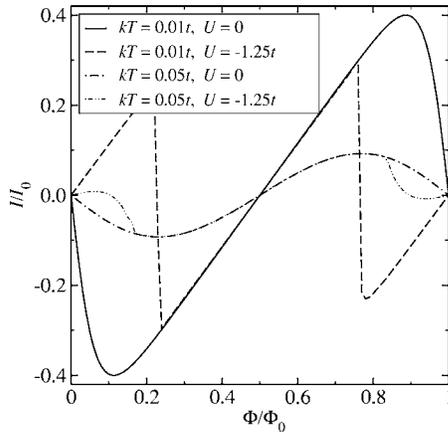


FIG. 7. Comparison of the persistent current in the presence and in the absence of the pairing potential for  $4 \times 30$  ring without impurities.

exactly exhibits  $\Phi_0/2$  periodicity. The flux dependence of the superconducting order parameter is similar to that of the pair susceptibility calculated for smaller rings with the help of the Lanczós method. However, in the latter case, regime A is much narrower due to a much smaller ring's size. It is well known that the mean-field approximation is inappropriate for low-dimensional systems. However, it seems that this simple approach correctly describes the persistent currents in small rings with weak local attraction.

In Fig. 7 we compare persistent currents in the normal and superconducting states. It is interesting that in regime B, the persistent current is the same in the presence and in the absence of the pairing interaction. Again, similar behavior has been obtained in the exact diagonalization study presented in the preceding section. In the upper panel of Fig. 4 one can see that for weak attraction the persistent current in the regime B hardly depends on  $U$ .

Finally, we investigate how the flux dependence of the order parameter depends on temperature in the case of nanoscopic rings. It has been well known since the famous experiment of Little and Park that properties of a superconducting thin film deposited on an insulating cylinder depend on an axial magnetic field.<sup>26</sup> In particular, the transition temperature is a periodic function of the magnetic flux with a period  $\Phi_0/2$ .<sup>27</sup> This experiment has been carried out for a macroscopic system. Figure 8 shows similar dependence for a small ring, where the finite-size effects are important. As one may expect also in this case the transition temperature is flux dependent. However, there is a visible deviation from the  $\Phi_0/2$  periodicity. Generally, in regime B the superconducting order parameter is less than in regime A. When the temperature increases, superconductivity first disappears in regime B and then in regime A. The same effect can be observed when the temperature is fixed but the pairing potential is reduced.<sup>32</sup> However, in contradiction to the macroscopic film, in the present case, the vanishing of superconductivity does not correspond to the vanishing of current. Comparing Figs. 7 and 8 one can note that the current also remains finite for  $\Delta_i=0$ .

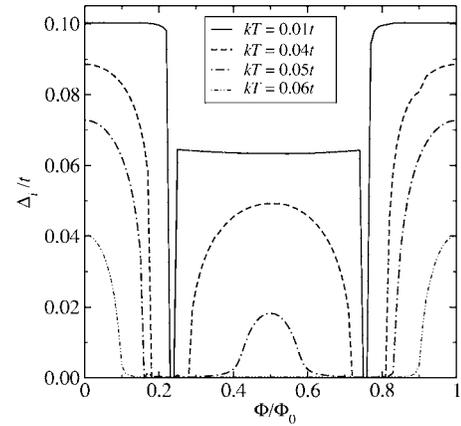


FIG. 8. The same as in Fig. 6 (for a  $4 \times 30$  ring), but at various temperatures.

### C. Impurities

In the case of small one-dimensional rings the presence of impurities significantly changes the persistent current. In the following, we show that the effect is very important in much larger rings as well. This holds also if the concentration of impurities is relatively low. In particular, similar to the one-dimensional case, the presence of a single impurity strongly reduces superconductivity. It originates from the pinning of the density wave, which competes with the superconductivity. In the case of many impurities, both the CDW and superconducting orders may coexist in the ring. However, the competition between CDW and superconductivity leads to a spatial separation of regions, where these orders dominate. The distribution of these regions is determined by the configuration of impurities. In the vicinity of impurities, the CDW order dominates. This enhancement of the CDW order is strongest, if the impurities are located in such a way, that the pinned density waves are in phase. Otherwise, the effect of impurities is much less important. This is a result of vanishing of the CDW order parameter somewhere in between the impurities. In the region of vanishing CDW order, the superconductivity is strongly enhanced. This effect is similar to that obtained for the vortex structure in Ref. 30, where the  $d$ -wave superconductivity competes with  $d$ -density waves.

Figure 9 shows the spatial distribution of the CDW and superconducting order parameters. The positions of the impurities are indicated by vertical arrows. In order to prove that the competition between these orders is responsible for their spatial distribution, we present also the sum of squares of the order parameters,  $\Psi_i = \sqrt{\Delta_i^2 + \Omega_i^2}$ . This quantity is almost constant over the whole ring (except for a very close vicinity of the impurities), which confirms our interpretation.

### V. CONCLUSIONS

We have presented numerical analysis of a nanoscopic ring pierced by an external magnetic flux. A large number of factors affecting the properties of the ring has been taken into account. In particular, we have rigorously treated the electron-electron interaction, demonstrating its role in the reduction of the persistent current. This effect, however, can be

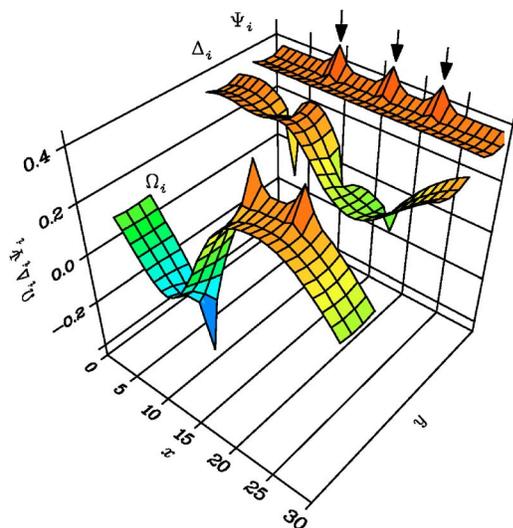


FIG. 9. (Color online) Spatial distribution of the CDW (the first stripe) and superconducting (the second stripe) order parameters,  $\Omega_i$  and  $\Delta_i$ , respectively, in  $4 \times 30$  ring for  $U = -4t$ ,  $W = -2t$  at  $kT = 0.01t$ . The third stripe shows  $\Psi_i = \sqrt{\Delta_i^2 + \Omega_i^2}$ . Arrows above the third stripe indicate the sites where impurities are located. For the sake of legibility, the ring has been cut and unfolded. The  $x$  and  $y$  axes show the position of lattice sites along the ring's circumference and width, respectively.

much less effective when impurities are introduced into the ring. It has been shown that the restoration of the persistent current occurs as a result of a reduction of density oscillations by the impurities. This is possible only for specific configurations of the impurities, namely, when the pinned density oscillations cancel each other out. Other configurations lead to an enhancement of the density oscillations and, simultaneously, to a reduction of the current.

The presence of the impurities strongly affects the superconducting properties of a nanoring as well. One may not expect a superconductivity in a system consisting of only several lattice sites, but the tendency toward the formation of a paired state can be analyzed. In particular, we have investigated a ring with an attractive on-site interaction. In order to estimate the strength of the pairing instability, we have calculated the maximal eigenvalue of the pair-susceptibility matrix. This is a gauge-invariant quantity that possesses the same space symmetry as the system and increases with the

amplitude of the pairing potential. We have shown that abrupt changes in the persistent current coincide with changes of the pair susceptibility. For very large pairing potential all electrons are paired and the flux dependence of the persistent current is the same as for free carriers of charge  $2e$ .

In the case of the attractive potential, the presence of impurities and their configuration are even more important than for a repulsive potential. It originates from the fact that for attractive interaction there is a competition between the CDW and superconductivity even in the absence of impurities, whereas for  $U > 0$  the density oscillations occur only in the vicinity of impurities. We have shown that impurities affect the superconducting properties indirectly, through an enhancement or a reduction of the CDW order. There is a single mechanism that determines how impurities affect both the persistent current for  $U > 0$  and the pair susceptibility for  $U < 0$ . Therefore, if a given impurity configuration leads to an enhancement of the persistent current for  $U > 0$ , the same configuration leads also to an enhancement of the pairing tendency for  $U < 0$ .

The exact diagonalization analysis has been supplemented by the Bogoliubov–de Gennes study of much larger rings of a finite width. Qualitatively both the approaches give similar results concerning the competition between superconductivity and CDW. In particular, the flux dependence of the pair susceptibility in the first case exactly corresponds to that of the pairing amplitude in the latter case. The role of impurities and their configuration in both cases are the same as well. Moreover, the BdG approach allowed us to investigate larger systems that exhibit bulk superconductivity and then, reducing their sizes, to trace how the properties change when entering the nanoregime. The mean-field approximation is generally inappropriate for low-dimensional systems. However, a comparison of the results obtained with the help of Lanczós and BdG methods indicates that the mean-field approach gives qualitatively correct results for the persistent currents in small rings with a weak pairing interaction.

To summarize, we have demonstrated how imperfectness modifies nanoscopic properties of small rings with electronic correlations. This investigation is important in connection with the recent developments in nanotechnology. In particular, it is possible to fabricate nanorings with arbitrary configuration of impurities and, in this way, to control the rings' properties.

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