Majorana modes in interacting systems identified by searching for local integrals of motion.

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There has recently been a substantial progress in methods of identifying local integrals of motion in interacting integrable models or in systems with many-body localization. We show that one of these approaches can be utilized for constructing local, conserved, Majorana fermions in systems with arbitrary many-body interaction. As a test case, we first investigate a non–interacting Kitaev model and demonstrate that this approach perfectly reproduces the standard results. Then, we discuss how the many-body interactions influence the spatial-structure and the lifetime of the Majorana modes. Finally, we determine the regime for which the information stored in the Majorana correlators is retained for arbitrary long times also at large temperatures. We show that it is included in the regime where the minimum–energy–states in even– and odd–parity sectors are degenerate, but in some cases is significantly smaller.

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Introduction. – Recently a lot of hopes have been pinned down on Majorana zero modes as building blocks of a quantum computer [1–5]. One of the systems where these modes were observed is a semiconductor nanowire with spin–orbit interaction coupled to s–wave superconductor [6–9]. It is known that in low–dimensional systems, Coulomb interactions are crucial and can drastically affect their properties [10, 11]: in the absence of the induced superconductivity in one dimension (1D) they lead to Luttinger liquids, that are very different from Fermi liquids in noninteracting systems. Additionally, in a strongly screened regime, the interactions can be driven by the metallic superconductor [12] or by the dielectric environment [13]. Interactions are important also for practical reasons: disorder is present in any semiconductor nanowire and the Majorana states are not completely immune against it [14–16]. Since atomic disorder acts like fluctuations of the local chemical potential, it is desirable to have a broad range of the chemical potential for which the Majorana modes exist. It has been demonstrated that this range increases if moderate interactions are present in the nanowire [17–19]. Interactions can also pin these states to zero–energy producing incompressible parameter regions where Majoranas remain insensitive to local perturbations [20].

There are several methods that can be used to study Majorana quasiparticles in interacting nanowires. They include, e.g., the density matrix renormalization group [21], Hartree-Fock theory, bosonisation, etc. (see Ref. [17] for their comparison). The problem, however, is how to distinguish the Majorana states in interacting systems from "accidental" zero–bias peaks, like the Andreev bound state [22] or the Kondo effect [23]. It has been shown that classifications of the topological phases in interacting and noninteracting systems are significantly different [24, 25], so it is not easy to use the same criteria. For example, in the topological phase the lowest–energy states with even and odd parity are degenerate and their many–body wave functions are related through some Majorana operator. These *necessary* requirements, however, in the presence of interactions do not guarantee the required robustness and long lifetime. In this Letter we formulate a *sufficient* condition which allows to identify Majorana operators, which are local and conserved also at large temperatures. Most importantly, our approach is based on solving a relatively small eigenproblem and can thus be easily implemented for arbitrary many-body interactions.

The General Method. – We considered Hamiltonian $\hat{H} = \sum_m E_m|m\rangle\langle m|$ and assume that the relevant degrees of freedom can be expressed in terms of the standard fermionic operators a_j and $a_j^\dagger$ or, equivalently, in terms of the Majorana fermions $\gamma_{2j} = a_j + a_j^\dagger$, $\gamma_{2j+1} = i(a_j^\dagger - a_j)$. Here, j includes all quantum numbers, e.g., the spin projection. In contrast to single–particle Hamiltonians, there is no unique scheme how to identify the presence of Majorana modes in interacting systems, although several hallmarks have been found [26–29]. Here, we focus on the most important (and potentially most useful) property of these modes that information stored in relevant correlators is preserved in time [30, 31]. In other words, we search for a particular combinations of the Majorana operators $\Gamma = \sum_i \alpha_i\gamma_i$ with real coefficients α_i such that Γ is conserved [32]. We assume normalization $\sum_i \alpha_i^2 = 1$ when $\Gamma^2 = 1$. The conservation of Γ can conveniently be studied by averaging this quantity over infinite time–window

$$\bar{\Gamma} = \lim_{\tau'\to\infty} \frac{1}{\tau'}\int_0^{\tau'} \mathrm{d}t e^{iHt}\Gamma e^{-iHt}, \tag{1}$$

$$= \lim_{\tau\to\infty}\sum_{m,n}\theta\left(\frac{1}{\tau} - |E_m - E_n|\right)\langle n|\Gamma|m\rangle \ \ |n\rangle\langle m|. \tag{2}$$

If this mode is strictly conserved then $\bar{\Gamma} = \Gamma$ however, in general, we will search for optimal choice of α_i when $\bar{\Gamma}$ is as *close* to Γ as possible. In order to quantify the proximity of two operators we use the usual (Hilbert-Schmidt) inner product $\langle\hat{A}\hat{B}\rangle = \mathrm{Tr}(\hat{A}\hat{B})/\mathrm{Tr}(\hat{1})$. The optimal choice of coefficients α_i corresponds to minimum of

$\langle(\Gamma-\bar{\Gamma})^2\rangle = 1-\langle\bar{\Gamma}^2\rangle$. The latter equality originates from the identity $\langle\Gamma\bar{\Gamma}\rangle = \langle\bar{\Gamma}\bar{\Gamma}\rangle$ (i.e., the time-averaging is an orthogonal projection) which can be easily shown, e.g., from Eq. (2). Consequently, the least decaying mode can be found from the optimization problem

$$\lambda = \max_{\{\alpha_i\}} \langle\bar{\Gamma}^2\rangle = \max_{\{\alpha_i\}} \langle\bar{\Gamma}\Gamma\rangle. \quad (3)$$

The physical meaning of λ comes from the observation that the scalar product $\langle ...\rangle$ formally represents thermal–averaging carried out for infinite temperature. Then, following Eq. (3), λ is the asymptotic value of the longest living autocorrelation function $\langle\Gamma(t)\Gamma\rangle$. If Γ is a strict integral of motion, then $\lambda = 1$. For $0 < \lambda < 1$ the information stored in the correlator $\langle\Gamma(t)\Gamma\rangle$ is partially retained for arbitrary long times (despite Γ in not strictly conserved), while this information is completely lost when $\lambda = 0$. The optimization problem can be farther simplified

$$\lambda = \max_{\{\alpha_i\}} \sum_{ij} \alpha_i \langle\bar{\gamma}_i\bar{\gamma}_j\rangle\alpha_j. \quad (4)$$

It becomes a standard eigenproblem for the (positive semi-definite) matrix $\langle\bar{\gamma}_i\bar{\gamma}_j\rangle$. Namely, λ is the largest eigenvalue of $\langle\bar{\gamma}_i\bar{\gamma}_j\rangle$, and α_j are components of the corresponding eigenvector. Essentially, all nonvanishing eigenvalues (whether degenerate or not) correspond to *independent* Majorana modes, whereby their independence follows from orthogonality of different eigenvectors and the identity $\langle\gamma_i\gamma_j\rangle = \delta_{ij}$.

The general idea behind this method is similar to other approach which has previously been used for identification of new integrals of motion in the Heisenberg model [33]. The latter approach targets operators which are conserved and local. Here, we single out operators which are conserved and, at the same time, are local Majorana fermions. The conservation follows from the time-averaging, i.e. $[\bar{\Gamma}, H] = 0$, whereas locality originates from that Γ is a linear combination of γ_i, each of them being supported on a single site only. Since we maximize the projection $\langle\bar{\Gamma}\Gamma\rangle$, the resulting operators retain the properties of both Γ and $\bar{\Gamma}$, i.e. they are local, conserved, Majorana fermions. The degeneracy of energy levels in subspaces with odd (E_o) and even (E_e) particle number is commonly used as a direct signature for Majorana modes. Indeed, such degeneracy allows to construct strict integrals of motion, $\sum_{E_e=E_o} b(E_e,E_o)|E_e\rangle\langle E_o|$+H.c. ("strong zero mode" in the present case [34–36]). However, in contrast to our approach, the degeneracy itself is insufficient to specify locality of such operator [37].

When studying systems with fixed boundary conditions it is utterly important, that the limit for the size of the system $L \to \infty$ precedes the limit for time $\tau \to \infty$, [38, 39]. Since numerical calculations can be carried out for finite systems only, τ in Eq. (2) should be kept large but finite until the finite–size scaling is accomplished. All the discussed properties of the correlation functions hold true also for finite τ [40, 41], even though it is not the case for finite τ' in Eq. (1).

Example. – As an example, we study a one–dimensional system of interacting, spinless fermions with hard-wall boundary conditions. The system is described by the Kitaev Hamiltonian [42] extended by the many–body interactions:

$$\hat{H} = -t_0 \sum_{i=1}^{L-1}\left(a^\dagger_{i+1}a_i + \text{H.c.}\right) + \Delta\sum_{i=1}^{L-1}\left(a^\dagger_{i+1}a^\dagger_i + \text{H.c.}\right) -\mu\sum_{i=1}^{L}\tilde{n}_i + V\sum_{i=1}^{L-1}\tilde{n}_i\tilde{n}_{i+1} + W\sum_{i=1}^{L-1}\tilde{n}_i\tilde{n}_{i+2}. \quad (5)$$

Here, t_0 refers to hopping amplitude, μ is a chemical potential, Δ is the superconducting gap and $\tilde{n}_i = a^\dagger_i a_i - \frac{1}{2}$. V and W are potentials of the first and second nearest-neighbor interactions. For simplicity, we use the dimensionless units by putting $\hbar = 1$ and t_0=1.

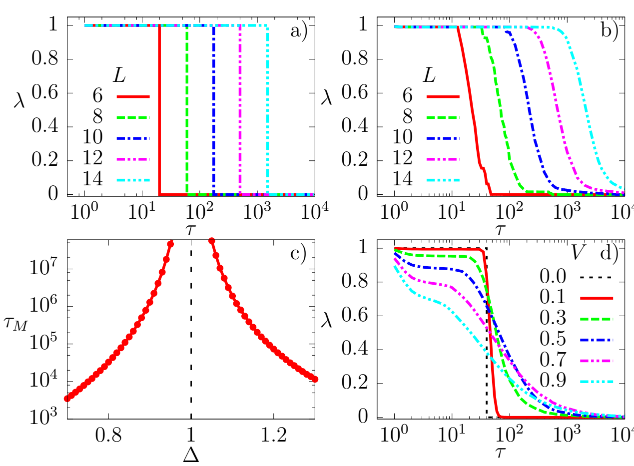

Figure 1. Results for systems without (a,c) and with (b,d) many-body interactions and $\mu = 0$. (a), (b) and (d) The Majorana autocorrelation function λ [see Eq. (3)] for: (a) $V = 0, \Delta = 0.5$; (b) $V = 0.2, \Delta = 0.5$; (d) $L = 12, \Delta = 0.3$. (c) Lifetime of Majorana modes for a finite noninteracting system of $L = 10$ sites.

Test for noninteracting systems. – Numerical implementation of our approach consists of three consecutive steps: *(i)* exact diagonalization of the Hamiltonian (5); *(ii)* numerical construction of time–averaged Majorana fermions $\bar{\gamma}_i$ as defined by Eq. (2) but for finite τ; *(iii)* construction and diagonalization of the matrix $K_{ij} = \langle\bar{\gamma}_i\bar{\gamma}_j\rangle$. Due to the orthogonality relation $\langle\bar{\gamma}_{2i}\bar{\gamma}_{2j+1}\rangle = 0$, one may separately study two cases $\Gamma^+ = \sum_i \alpha^+_i\gamma_{2i}$ and $\Gamma^- = \sum_i \alpha^-_i\gamma_{2i+1}$, whereby now the index i enumerates the lattice sites. In the rest of this work we discuss two most stable modes (one in each sector Γ^+ and Γ^-). All other eigenvalues of the matrix K are much smaller and

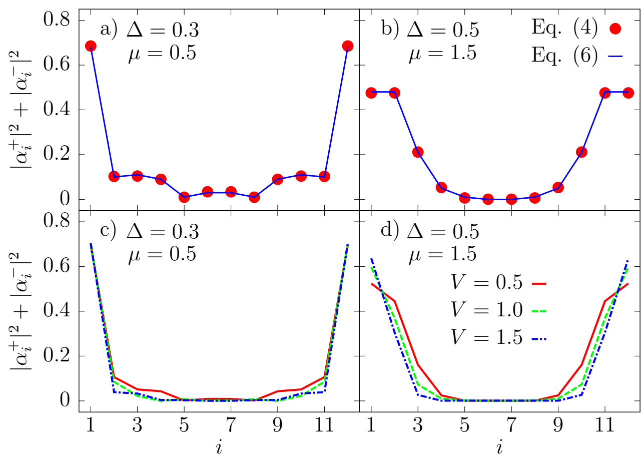

Figure 2. Spatial structure of Majorana modes, $\Gamma^+ = \sum_i \alpha_i^+ \gamma_{2i}$ and $\Gamma^- = \sum_i \alpha_i^- \gamma_{2i+1}$. a) and b) Rescaled local density of states at energy $E = 0$ for noninteracting system [$V = 0$, Eq. (6)] (solid line) compared with solution of Eq. (4) (points). c) and d) Results for $V \neq 0$ from Eq. (4).

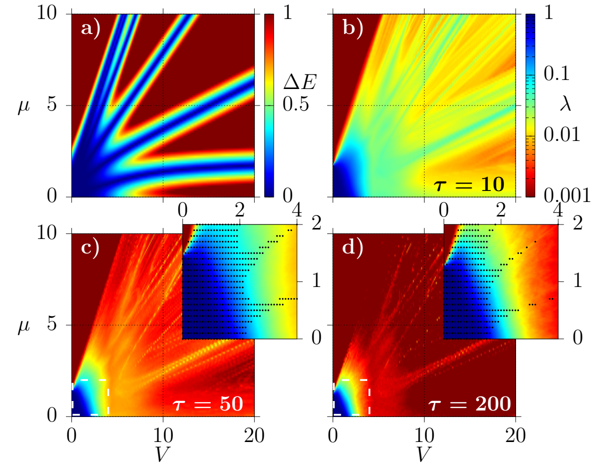

Figure 3. Results for $L = 12$ and $\Delta = 1$. a) Difference between the ground state energies for different parities of fermions, ΔE. b)-d) The Majorana autocorrelation function λ for various times τ. Insets show magnified details. Points mark parameters for which the ground state $\Delta E \leqslant 1/\tau$.

vanish in the thermodynamic limit (not shown), in agreement with common knowledge that the model (5) may host at most two Majorana modes.

The complexity of our approach is independent of whether the many-body interactions are present or not, hence the method can be best tested by investigating a noninteracting system with $V = W = 0$. Fig. 1a shows τ–dependence of λ (see Eq. 3) for the most stable Majorana mode Γ^+. Results for Γ^- are exactly the same. One may introduce the lifetime of the Majorana modes, τ_M, corresponding to the vertical sections of curves shown in the latter figure. Fig. 1c shows that for finite system τ_M is finite as well, despite the absence of the many-body scattering. The only exception concerns $|\Delta| = 1$ when $\tau_M \to \infty$ for arbitrary L. Otherwise, τ_M increases exponentially with L, as follows from the equal spacing of the vertical sections in Fig. 1a. The latter result clearly illustrates importance of the correct order of limits: $\lim_{\tau\to\infty} \lim_{L\to\infty} \lambda = 1$, i.e. the Majorana modes are strictly conserved in the thermodynamic limit, while $\lim_{L\to\infty} \lim_{\tau\to\infty} \lambda = 0$. All the obtained results remain in agreement with the well established properties of the Majorana modes in noninteracting case, see e.g. [43].

In order to carry out a quantitative test, we have also calculated the local density of states at zero energy for the noninteracting Hamiltonian

$$\rho_i(E = 0) = -\frac{1}{\pi} \mathrm{Im}\, G_{ii}(E = 0), \quad G(E) = (E - \hat{H} + i\eta)^{-1}, \tag{6}$$

where $\hat{H}$ is given by Eq. (5) but with $V = W = 0$. In Figs. 2a and 2b rescaled $\rho_i(E = 0)$ is compared with the spatial density of the Majorana fermions contributing to both Majorana modes, $|\alpha_i^+|^2 + |\alpha_i^-|^2$. Perfect agreement between both methods illustrates accuracy of the approach derived in this work.

Systems with many-body interactions. – It is important to stress the differences between the present approach and the previous studies on the Majorana modes in systems with many body-interactions. The main distinction is that we check a single condition which is *sufficient* for the presence of local, conserved Majorana fermions, whereas one usually formulates and independently tests several *necessary* conditions. While the other studies mostly focus on the ground state properties, in our approach the Hilbert-Schmidt inner product naturally introduces averaging over the Gibbs ensemble at infinite temperature, similarly to Refs. [30, 31]. Therefore, we recognize regime where the Majorana modes are robust against thermal excitations. The inner product may be modified in such a way that it becomes relevant for finite temperatures, e.g. see Ref. [40]. We have numerically studied selected cases also for $k_B T = 10$ and found that the liftetime of the Majora modes is slightly larger than at $T \to \infty$ (not shown).

All results in the main text will be shown for $W = V/2$, whereas the commonly studied case $W = 0$ (which contains some peculiar features) is discussed in the Supplemental Material [44]. Results in Figs. 1b and 1d show the most stable Majorana autocorrelation function [Eq. (3)] in the presence of weak to moderate interactions. Similarly to the noninteracting case (fig. 1a), the position of the steep sections of $\lambda(\tau)$ increases exponentially with the system size indicating that $\lim_{\tau\to\infty} \lim_{L\to\infty} \lambda \simeq 1$ but, in contrast to noninteracting systems, $\lambda < 1$. In the supplemental material [44], we show that the latter inequality seems to be generic for systems with many-body interactions. It implies that local operator Γ is not a strict integral of motion. It has large projection on other

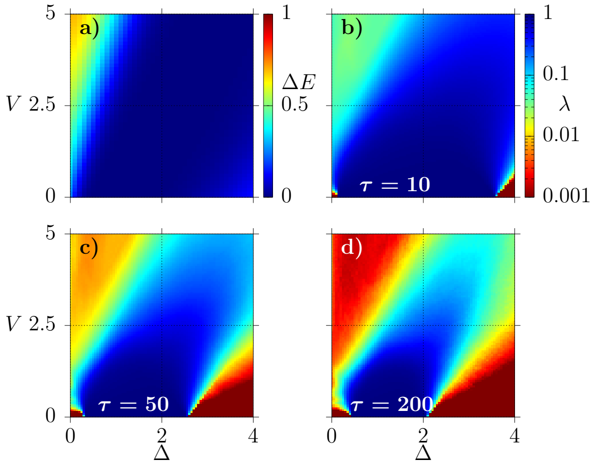

Figure 4. The same as in Fig. 3 but for $\mu = 0$ and various Δ.

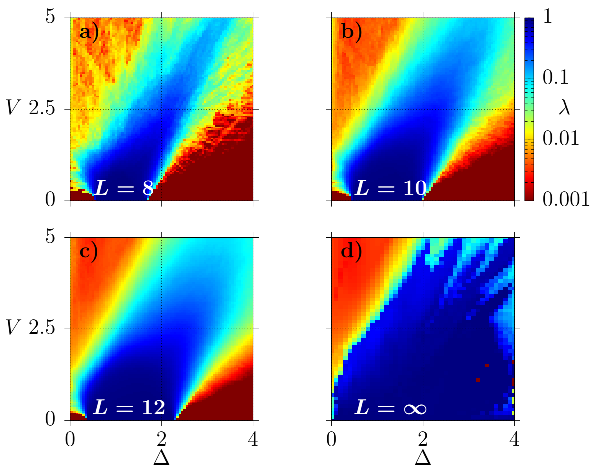

Figure 5. The same as in Figs. 4b–4d but for $\tau = 100$ and various system sizes L.

strict integral of motion $\lim_{\tau\to\infty} \bar{\Gamma}$, which however may be nonlocal. This large projection implies that $\lim_{\tau\to\infty} \bar{\Gamma}$ is a quasilocal integral of motion [33, 40] which must have similar form to that discussed in Ref. [30].

For finite systems, the many-body interactions may extend the time–scale in which the correlator $\langle \Gamma(t)\Gamma \rangle$ is large. Interestingly, this extension can exceed one order of magnitude, as shown in Fig. 1d. Figures 2c and 2d explain the origin of this extension. They show how the many–body interactions modify the spatial structure of the Majorana modes. There are two modes which vanish exponentially outside of compact regions in the real-space located at the edges of system. Despite the exponential decay, these two modes still do overlap and this overlap is responsible for a finite-lifetime of the Majorana modes in a noninteracting system with $L < \infty$. Then, the many–body interactions push these modes further towards the edges of the system (see Figs. 2c and 2d), reducing the overlap between them and, in this way, increasing their lifetime. This mechanism holds true as long as the interactions are not too strong, when the Majorana modes eventually disappear.

In Fig. 3 we compare our approach with a *necessary* condition which is a common criterion for the presence of the Majorana modes. The latter requirement concerns degeneracy of the ground states obtained for systems with odd and even number of fermions. As shown in Fig. 3a, this requirement is satisfied within an extended 2D region in the μ–V plane as well as along a few lines going out of it. Our results show that the Majorana modes indeed exist for very long times ($\tau > 200$) not only in the ground state but essentially in the entire energy spectrum (see insets in Figs. 3c and 3d). However, they exist only within the former 2D region, while they don't appear along the lines. As explicitly shown in the supplemental material [44], these lines separate regimes where the (ground state) occupations are close to the consecutive integers $0, 1, ..., L$. Therefore, the degeneracy of ground states along these lines results merely from level crossings and is unrelated to the Majorana modes. The latter example illustrates advantage of using our method which directly tests the Majorana modes. We confirm that moderate many-body interaction extends the range of μ where Majorana modes are present [17].

In Fig. 4 we show similar results but for $\mu = 0$ and various magnitudes of the superconducting gap Δ. In this case an exact solution exists for $\Delta = 1$ and $W = 0$ [45, 46] which compared to results in the Supplemental Material shows that vanishing of ΔE indeed signals the presence of the topological phase. For large τ and $\Delta \gg 1$, the Majorana modes seem to be absent even for very weak many-body interaction. However, it is the finite–size effect that again shows, how important is the correct order of limits for time and the system size. Therefore, in Fig. 5 we set $\tau = 100$ and show the Majorana autocorrelation function for various L together with results extrapolated to $L \to \infty$. The details of extrapolation and results for $\lim_{\tau\to\infty} \lim_{\tau\to\infty} \lambda(\tau)$ are shown in the supplemental material [44]. It is important to stress that after appropriate finite–size scaling one finds that the Majorana modes exist in the regime of parameters which is only slightly smaller that the regime determined from the degeneracy of the ground states (compare Figs. 4a and 5d). Additionally, results in 5a – 5c show that λ increases with L. Therefore, even if the finite–size scaling is too tedious, numerical results obtained for certain L can be used as a lower bound for the existence of the Majorana modes.

Conclusions. – We have proposed an approach for finding local Majorana modes with the longest lifetime. Since the numerical complexity consists just in solving a standard eigenproblem, it can be easily implemented for arbitrary many–body interaction. We have found that even at elevated temperatures the lifetime of these modes is long enough so that they may effectively be used to store the information. The parameter regime where the long–living Majorana states exist is included, but smaller than the topological regime defined by degeneracy of the minimum–energy–states in even– and odd–parity sectors. It means that not all topological states are sufficiently protected to be useful in, e.q., quantum computing. Our results also suggest that in systems with many-body interactions the strictly local Majorana operators are not strict integrals of motion, however, their autocorrelation function remains large for arbitrary long times. This, in turn, implies the presence of quasilocal conserved operators.

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Supplemental Material

In the Supplemental Material we discuss results for a system where the many–body interactions are restricted to the neighboring lattice sites only. We also present the details of the finite–size scaling.

RESULTS FOR A SYSTEM WITH NEAREST NEIGHBOR INTERACTION

Numerical results presented in the main text have been obtained for repulsive interaction between the first (V) and the second nearest neighbors (W). Here, we discuss the same quantities but for the commonly studied case with $W = 0$ which, however, contains some peculiar features.

We start with the difference of the ground state energies, ΔE, obtained for systems with odd and even number of fermions. Vanishing of ΔE is *necessary* for the onset of Majorana modes. This quantity is shown in Figs. S1a and S1b for $W = 0$ and $W = V/2$, respectively. The former figure accurately reproduces results presented in Ref. [26]. Here one may identify two regions where ΔE is small (vanishes in the thermodynamic limit) and the Majorana modes may exist: (1) small 2D area around $V = 0$, $\mu = 0$ and (2) a number of narrow stripes that extend from region (1) to large–V and large–μ. While the internal structure of region (2) is hardly visible for $W = 0$, it becomes very clear for $W = V/2$, as shown in Fig. S1b. It is composed of stright lines and their total number exactly equals the size of the system, whereby $L/2$ lines are explicitly visible in the figures, while remaining $L/2$ lines exist for $\mu < 0$. The physical origin of these lines is explained in Figs. S1c and S1d where we show average particle number in the ground state, $\langle N \rangle$. We also mark the parameters for which ΔE is small. One can see that lines separate regimes where the ground state occupation is close to consecutive integers $0, 1, ..., L$. In the adjoining regimes, these integers have opposite parity, hence the borderline between these regimes corresponds to degenerate ground states obtained in sectors with odd and even particle numbers. However, such lines exists independently of the pairing term, i.e., also in topologically trivial phases, and are thus unrelated to the local Majorana modes.

Figures S2, S3 and S4 show respectively the same phase diagrams as Figs. 3, 4 and 5 in the main text but for $W = 0$. These phase diagrams are constructed from the Majorana autocorrelation function λ. On the one hand, these plots clearly show that the qualitative results discussed in the main text are generic, i.e., they are independent of a specific choice of the many–body interaction. The main message is that the long–living Majorana modes exist not only in the ground state but within the whole energy spectrum for moderate interactions, while

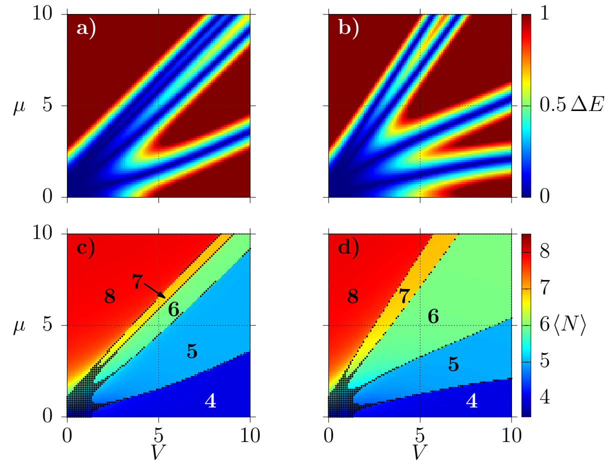

Figure S1. Results for $L = 8$, $\Delta = 1$ and $W = 0$ (a,c) or $W = V/2$ (b,d). a) and b) Difference between the ground state energies obtained for different parities of fermions, ΔE. c) and d) Average occupation of fermions in the ground state. Points mark parameters for which $\Delta E < 0.02$.

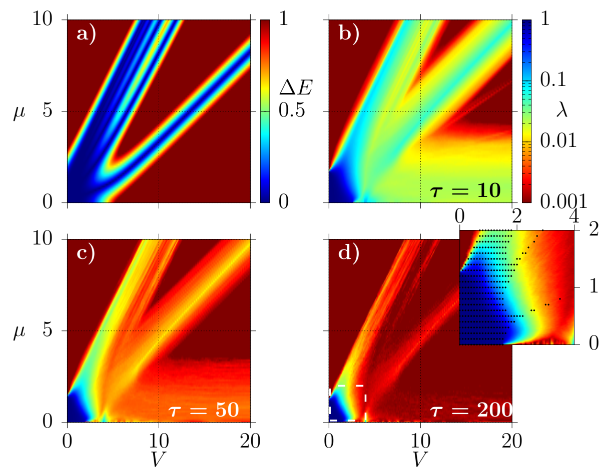

Figure S2. The same as Fig. 3 in the main text but for $W = 0$.

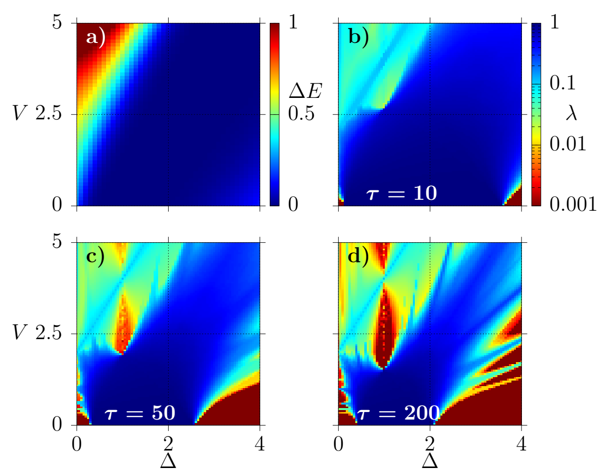

Figure S3. The same as Fig. 4 in the main text but for $W = 0$.

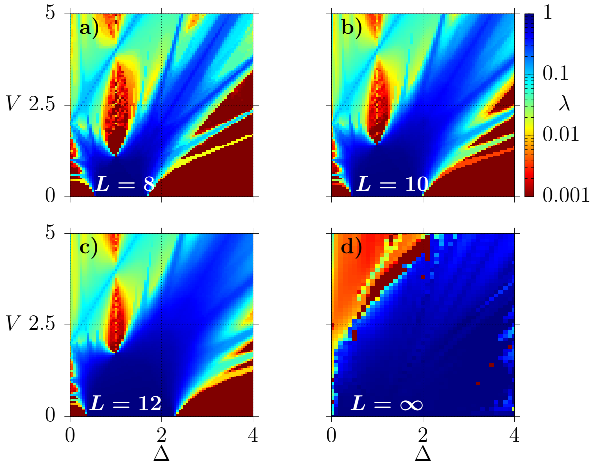

Figure S4. The same as Fig. 5 in the main text but for $W = 0$.

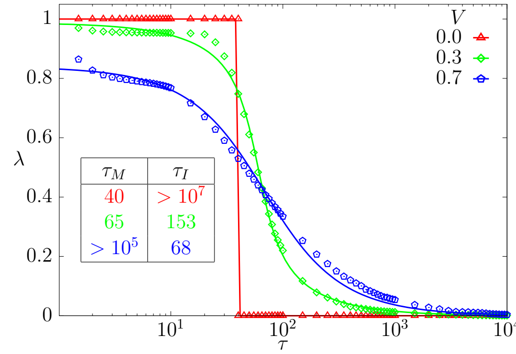

Figure S5. The Majorana autocorrelation function λ for various times τ. Points show numerical results for $L = 12$, $\Delta = 0.3$, $\mu = 0$ and $W = V/2$ while continuous lines show best fits, given by Eq. (S4). The scattering times, τ_M and τ_I, are shown in the table.

weak interaction may even expand the range of the chemical potential where these modes exist see the inset in Fig. S2d). The Majorana modes exist in the regime of parameters which is only slightly smaller that the regime determined from the degeneracy of the ground states (compare Figs. S3a and S4d) after excluding the lines which were discussed in the preceding paragraph. On the other hand, the phase diagrams for $W = 0$ include rather complicated structure of lines where Majorana modes are particularly robust, see e.g., Figs. S3d and S4a-S4c. However, such structures are not generic because they don't show up for $W \neq 0$.

DETAILS OF THE FINITE-SIZE SCALING

As argued in the main text, it is utterly important for systems with hard–wall boundary conditions that the thermodynamic limit $L \to \infty$ precedes the limit $\tau \to \infty$. It means that the correct finite–size (FS) scaling should be carried out not for a single quantity but for the τ–dependent Majorana autocorrelation function $\lambda(\tau)$. In order to efficiently perform such scaling, we first fit $\lambda(\tau)$ for a given system length L and then carry out the FS scaling for the fitting parameters. It is convenient to start from a standard time–dependent correlation function

$$\langle \Gamma(t)\Gamma\rangle = \sum_{m,n} \exp[it(E_m - E_n)] |\langle n|\Gamma|m\rangle|^2. \quad \text{(S1)}$$

It differs from the τ–dependent autocorrelation functions $\langle \bar{\Gamma}\Gamma\rangle$ which utilizes time–averaging introduced in Eq. (2) in the main text. However, the relation between both functions can be easily established

$$\begin{aligned}\langle \bar{\Gamma}\Gamma\rangle &= \sum_{m,n} \theta\left(\frac{1}{\tau} - |E_m - E_n|\right) |\langle n|\Gamma|m\rangle|^2 \\ &= \sum_{m,n} |\langle n|\Gamma|m\rangle|^2 \int_{-1/\tau}^{1/\tau} \mathrm{d}\omega\, \delta(\omega + E_m - E_n) \\ &= \frac{1}{2\pi} \int_{-1/\tau}^{1/\tau} \mathrm{d}\omega \int_{-\infty}^{\infty} \mathrm{d}t e^{i\omega t} \langle \Gamma(t)\Gamma\rangle. \quad \text{(S2)}\end{aligned}$$

One immediately finds the limit

$$\lim_{\tau\to\infty} \langle \bar{\Gamma}\Gamma\rangle = \sum_{m,n:E_m=E_n} |\langle n|\Gamma|m\rangle|^2.$$

This limit is exactly equal to the steady–state part of $\langle \Gamma(t)\Gamma\rangle$ [see Eq. (S1)] which survives for arbitrarily large t. For finite τ, $\langle \bar{\Gamma}\Gamma\rangle$ represents the integrated low–frequency part of the Fourier transform of $\langle \Gamma(t)\Gamma\rangle$, as follows from the last line in Eq. (S2). We use $\langle \bar{\Gamma}\Gamma\rangle$ instead of $\langle \Gamma(t)\Gamma\rangle$ because only the former function is monotonic. It filters out the oscillations of $\langle \Gamma(t)\Gamma\rangle$ but retains essential information about the asymptotic long–time behavior [41]. But most importantly, the efficiency of our method follows from that the specific time–averaging, $\bar{\Gamma}$, is an orthogonal projection also for $\tau < \infty$, i.e., $\langle \bar{\Gamma}\Gamma\rangle = \langle \bar{\Gamma}\bar{\Gamma}\rangle$.

In order to find the relevant fitting function for $\lambda(\tau)$, we notice that generic many–body interactions are expected to cause exponential decay of correlations functions. It holds true also for perturbed integrable systems [40]. For $\langle \Gamma(t)\Gamma\rangle = \exp(-t/\tau_I)$ one obtains from Eq. (S2)

$$\langle \bar{\Gamma}\Gamma\rangle = \frac{2}{\pi} \operatorname{arc\,tg}\left(\frac{\tau_I}{\tau}\right). \quad \text{(S3)}$$

However, such form is too simple to account for *finite* noninteracting systems, where the Majorana lifetime, τ_M, is limited by the overlap of Majorana modes at two ends of the chain. Without interactions, the Majorana autocorrelation function is a step function, as shown in

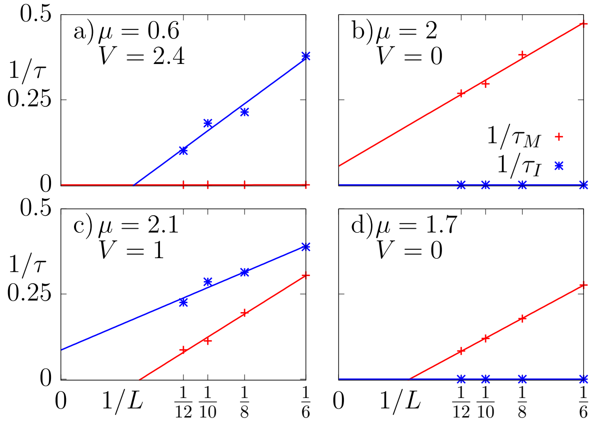

Figure S6. Representative examples for the finite–size scaling of scattering times, τ_I and τ_M for $\Delta = 1$, $W = 0$.

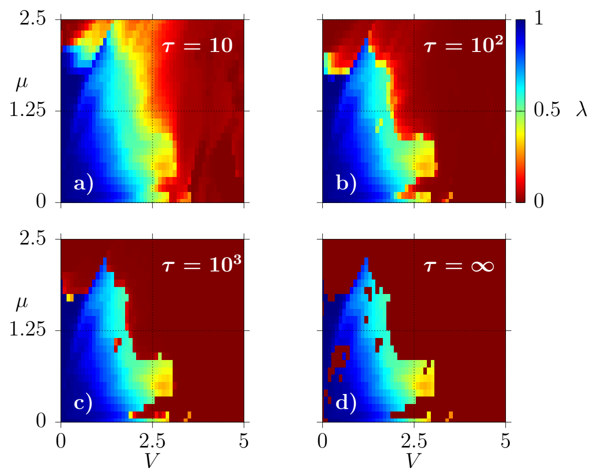

Figure S7. Extrapolated Majorana autocorrelation function, $\lim_{L\to\infty} \lambda$ for $\Delta = 1$, $W = 0$ various times τ.

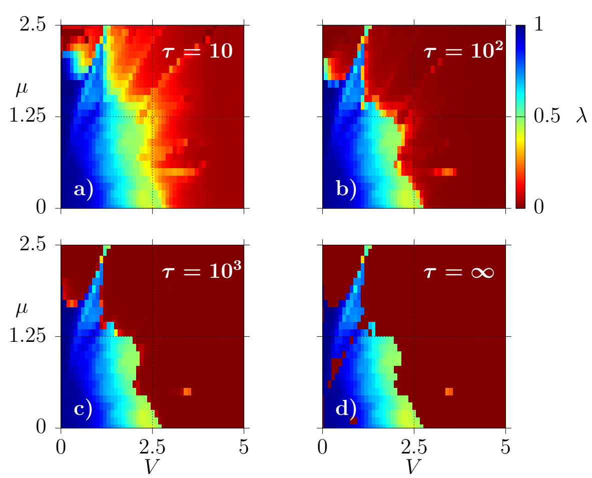

Figure S8. The same as in Fig. S7 but for $W = V/2$.

Fig. 1a in the main text. In order to accommodate this mechanism we have modified Eq. (S3)

$$\lambda_{\rm fit}(\tau) = \frac{C}{\pi}\left[\operatorname{arc\,tg}\left(\frac{\tau_{\rm I}}{\tau} - \frac{\tau_{\rm I}}{\tau_{\rm M}}\right) + \operatorname{arc\,tg}\left(\frac{\tau_{\rm I}}{\tau} + \frac{\tau_{\rm I}}{\tau_{\rm M}}\right)\right]. \tag{S4}$$

The fitting function contains three parameters: C, $\tau_{\rm I}$ and $\tau_{\rm M}$. Here, $1/\tau_{\rm M}$ and $1/\tau_{\rm I}$ represent scattering rates due to the overlap of two Majorana modes and due to the many–body interactions, respectively. If the relaxation is dominated by the many–body interactions, then $\lim_{\tau_{\rm M}\to\infty} \lambda_{\rm fit}(\tau) = \frac{2C}{\pi} \operatorname{arc\,tg}\left(\frac{\tau_{\rm I}}{\tau}\right)$. However, if the relaxation is due to overlap of two Majorana modes, then $\lim_{\tau_{\rm I}\to\infty} \lambda_{\rm fit}(\tau) = C\theta(\tau_{\rm M} - \tau)$, in agreement with Fig. 1a in the main text. Figure S5 shows that $\lambda(\tau)$ may be well fitted by Eq. (S4) also for intermediate cases, when both scattering mechanisms are important.

In Fig. S6 we show a few representative examples for the finite size scaling of the scattering rates $1/\tau_{\rm I}$ and $1/\tau_{\rm M}$. One is mostly interested in the case when both scattering rates vanish in the thermodynamic limit (see Figs. S6a, S6d) and $\lim_{\tau\to\infty} \lim_{L\to\infty} \lambda(\tau) = \lim_{L\to\infty} C$. Otherwise, $\lim_{\tau\to\infty} \lim_{L\to\infty} \lambda(\tau) = 0$ if one of the scattering times remain finite. In noninteracting system, the Majorana modes exist for $|\mu| < 2$ what is correctly reproduced by our approach as shown in Figs. S6b and S6d.

After the FS scaling has been accomplished, one may study (approximate) results for the Majorana autocorrelation function in the thermodynamic limit presented in Figs. S7 and S8. These plots show $\lambda_{\rm fit}(\tau)$ where all fitting parameters are replaced by their extrapolated values. Such procedure unavoidably introduces errors. Consequently the irregular shape of the regime with Majorana modes most probably arises as a numerical artefacts. Nevertheless, it is rather evident that the information stored in the Majorana autocorrelation functions is at least partially retained for arbitrarily long times also for rather strong interactions $V \lesssim 2$. However, if the Majorana modes are strict integrals of motion then $\lim_{\tau\to\infty} \lim_{L\to\infty} \lambda(\tau) = 1$. Numerical results shown in Figs. S7 and S8 strongly suggest that in the presence of many–body interactions, the latter limit is always smaller than unity, even though it may be very close to this value. As argued in the main text, this implies the presence of quasilocal strictly conserved operators $\lim_{\tau\to\infty} \bar{\Gamma}$, which have large projection on strictly local Majorana modes, Γ.

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